Analysis of Fractional-Order Prey - Predator Model with Harvesting

A. George Maria Selvam[1] and S. Britto Jacob[2]

[1,2]*Department of Mathematics, Sacred Heart College (Autonomous),*
Tirupattur, Vellore, S. India – 635 601.

Abstract— ***This work analyzes the dynamical behavior of harvesting in a discrete fractional order prey - predator model. Utilizing a discretization process to the fractional order model, we obtain its discrete version. Existence of fixed points is established and a detailed analysis of the stability of fixed points is carried out with the linearization process and the impact of harvesting on the ecosystem is discussed. To verify the analytical results, numerical evidence for different sets of biologically feasible parameter values is provided. Time series, phase portrait and bifurcation diagrams are plotted to study the stability of the model. Analysis of sensitivity to the initial conditions of the system is also performed.***

Keywords—***Fractional order, prey - predator model, harvesting, stability, bifurcation.***

1. Introduction

Prey - predator model is one of the basic relationships between different species in an ecosystem. In order to interpret the phenomena in nature and provide a guide to analyze, various types of predator-prey models are described by systems described by differential equations, difference equations, partial differential equations and stochastic differential equations. Both mathematical modeling and simulation of the ecological systems have received a great deal of attention in the last few decades [3, 6]. Many mathematical models [1, 5, 7] have been constructed based on more realistic biological assumptions.

2. Model Description

Consider the prey-predator system with constant effort harvesting which is described by the following fractional differential equations

$$\begin{aligned} D^{\alpha} x(t) &= rx(t)(1-x(t)) - \frac{bx(t)y(t)}{1+x(t)} - mx(t) \\ D^{\alpha} y(t) &= \frac{cx(t)y(t)}{1+x(t)} - dy(t) \end{aligned} \qquad (1)$$

where $x(t)$ and $y(t)$ represent the density of the prey and predator at time t respectively. The parameter m measures the effort being spent by a harvesting agency and all the parameters r, b, c, d are positive and α is the fractional order such that $\alpha \in (0,1]$. Here the prey is continuously being harvested at a linear function rate by a harvesting agency. The harvesting activity does not affect the predator population directly. It is obvious that the harvesting activity does reduce the predator population indirectly by reducing the availability of the prey to the predator.

The fixed points of the system (1) are :

- Trivial point $F_0 = (0,0)$, Axial point $F_1 = \left(\frac{r-m}{r}, 0\right)$
- Interior point $F_2(x^*, y^*) = \left(\frac{d}{c-d}, \frac{c(cr-cm+dm-2dr)}{b(c-d)^2}\right)$

Now, applying the discretization process [2], the discrete version of the fractional order prey- predator system (1) is:

$$
\begin{aligned}
x_{n+1} &= x_n + \frac{s^\alpha}{\alpha\Gamma(\alpha)}\left(rx(1-x) - \frac{bxy}{1+x} - mx \right) \\
y_{n+1} &= y_n + \frac{s^\alpha}{\alpha\Gamma(\alpha)}\left(\frac{cxy}{1+x} - dy \right)
\end{aligned} \tag{2}
$$

3. Mathematical Analysis of the System

This section investigates the dynamics of the discretized fractional - order prey - predator model described by (2). The system matrix of the model (2) evaluated a fixed point (x^*, y^*) is

$$
J(x^*, y^*) = \begin{bmatrix} 1+\frac{s^\alpha}{\alpha\Gamma(\alpha)}\left(r - 2rx^* - \frac{by^*}{(1+x^*)^2} - m \right) & -\frac{s^\alpha}{\alpha\Gamma(\alpha)}\left(\frac{bx^*}{(1+x^*)} \right) \\ \frac{s^\alpha}{\alpha\Gamma(\alpha)}\left(\frac{cy^*}{(1+x^*)^2} \right) & 1+\frac{s^\alpha}{\alpha\Gamma(\alpha)}\left(\frac{cx^*}{(1+x^*)} - d \right) \end{bmatrix}. \tag{3}
$$

Theorem 1. *If* $0 < s < \sqrt[\alpha]{\frac{2\alpha\Gamma(\alpha)}{r-m}}$ *and* $0 < s < \sqrt[\alpha]{\frac{2\alpha\Gamma(\alpha)}{-d}}$ *then* F_0 *is stable. If* $s > \sqrt[\alpha]{\frac{2\alpha\Gamma(\alpha)}{r-m}}$ *and* $s > \sqrt[\alpha]{\frac{2\alpha\Gamma(\alpha)}{-d}}$ *then* F_0 *is unstable.*

Proof: The system matrix at F_0 is given by $J(F_0) = \begin{bmatrix} 1+\frac{s^\alpha}{\alpha\Gamma(\alpha)}(r-m) & 0 \\ 0 & 1+\frac{s^\alpha}{\alpha\Gamma(\alpha)}(-d) \end{bmatrix}.$

The eigenvalues are $\lambda_1 = 1+\frac{s^\alpha}{\alpha\Gamma(\alpha)}(r-m)$ and $\lambda_2 = 1+\frac{s^\alpha}{\alpha\Gamma(\alpha)}(-d)$.

Hence the fixed point F_0 is stable when $0 < s < \sqrt[\alpha]{\frac{2\alpha\Gamma(\alpha)}{r-m}}$ and $0 < s < \sqrt[\alpha]{\frac{2\alpha\Gamma(\alpha)}{-d}}$. Then F_0 is unstable if $s > \sqrt[\alpha]{\frac{2\alpha\Gamma(\alpha)}{r-m}}$ and $s > \sqrt[\alpha]{\frac{2\alpha\Gamma(\alpha)}{-d}}$.

Theroem 2. *If* $0 < s < \sqrt[\alpha]{\frac{2\alpha\Gamma(\alpha)}{m-r}}$ *and* $0 < s < \sqrt[\alpha]{\frac{(4r-2m)\alpha\Gamma(\alpha)}{r(c-2d)+m(d-c)}}$ *then* F_1 *is stable. If* $s > \sqrt[\alpha]{\frac{2\alpha\Gamma(\alpha)}{m-r}}$ *and* $s > \sqrt[\alpha]{\frac{(4r-2m)\alpha\Gamma(\alpha)}{r(c-2d)+m(d-c)}}$ *then* F_1 *is unstable.*

Proof: The system matrix at F_1 is $J(F_1) = \begin{bmatrix} 1+\frac{s^\alpha}{\alpha\Gamma(\alpha)}\left(r - 2r\left(\frac{-m+r}{r}\right) - m \right) & -\frac{s^\alpha}{\alpha\Gamma(\alpha)}\left(\frac{bm-r}{-m} \right) \\ 0 & 1+\frac{s^\alpha}{\alpha\Gamma(\alpha)}\left(\frac{c\left(\frac{r-m}{r}\right)}{1+\left(\frac{r-m}{r}\right)} - d \right) \end{bmatrix}.$

The eigenvalues are $\lambda_1 = 1+\frac{s^\alpha}{\alpha\Gamma(\alpha)}(m-r)$ and $\lambda_2 = 1+\frac{s^\alpha}{\alpha\Gamma(\alpha)}\left(\frac{r(c-2d)+m(d-c)}{2r-m} \right)$.

Hence the fixed point F_1 is stable when $0 < s < \sqrt[\alpha]{\frac{2\alpha\Gamma(\alpha)}{m-r}}$ and $0 < s < \sqrt[\alpha]{\frac{(4r-2m)\alpha\Gamma(\alpha)}{r(c-2d)+m(d-c)}}$.

Then F_1 is unstable if $s > \sqrt[\alpha]{\dfrac{2\alpha\Gamma(\alpha)}{m-r}}$ and $s > \sqrt[\alpha]{\dfrac{(4r-2m)\alpha\Gamma(\alpha)}{r(c-2d)+m(d-c)}}$.

Theroem 3. *The positive fixed point* (x^*, y^*) *of the prey - predator system (2) is stable if and only if* $\beta < \left(\dfrac{\alpha\Gamma(\alpha)}{s^\alpha}\right)\left[2+\left(\dfrac{s^\alpha}{\alpha\Gamma(\alpha)}\right)^2\left(\dfrac{c(\delta)}{1+\dfrac{d}{c-d}}\right)\left(\dfrac{bd}{c}\right)\right]$ *and* $b > \left(\dfrac{\alpha\Gamma(\alpha)}{s^\alpha}\right)^2\left[\dfrac{c}{(\delta)d(c-d)^2}\right]$.

Proof. The system matrix evaluated at the fixed point F_2 has the form

$$J(F_2) = \begin{bmatrix} 1+\dfrac{s^\alpha}{\alpha\Gamma(\alpha)}(\beta) & -\dfrac{s^\alpha}{\alpha\Gamma(\alpha)}\left(\dfrac{bd}{c}\right) \\ \dfrac{s_\alpha}{\alpha\Gamma(\alpha)}\left(\dfrac{c(\delta)}{\left(1+\dfrac{d}{c-d}\right)^2}\right) & 0 \end{bmatrix}.$$

where $\beta = \dfrac{rd(bcd-2c^2-bd^2)+md(2d^2-3cd+d^2)}{(c-d)(c^2+d^2-2cd+d^2)}$ and $\delta = \dfrac{c[cr-cm+dm-2dr]}{b(c-d)^2}$

The trace and determinant of the matrix $J(F_2)$ are given by

$$TrJ(F_2) = 1+\frac{s^\alpha}{\alpha\Gamma(\alpha)}(\beta) \text{ and } DetJ(F_2) = \left(\frac{s^\alpha}{\alpha\Gamma(\alpha)}\right)^2\left(\frac{c(\delta)}{\left(1+\dfrac{d}{c-d}\right)^2}\right)\left(\frac{bd}{c}\right)$$

According to the *Jury conditions*[4], we must have that $P(1) > 0$ holds if and only if $r > 0$.

Similarly $P(-1) > 0$ and $DetF_2 < 1$ holds if and only if $\beta < \left(\dfrac{\alpha\Gamma(\alpha)}{s^\alpha}\right)\left[2+\left(\dfrac{s^\alpha}{\alpha\Gamma(\alpha)}\right)^2\left(\dfrac{c(\delta)}{1+\dfrac{d}{c-d}}\right)\left(\dfrac{bd}{c}\right)\right]$ and $b > \left(\dfrac{\alpha\Gamma(\alpha)}{s^\alpha}\right)^2\left[\dfrac{c}{(\delta)d(c-d)^2}\right]$ respectively.

4. Numerical Illustration

For the values $\alpha = 0.9, r = 1.29, b = 1.9, c = 1.7, d = 0.3, s = 0.1$ and $m = 0.09$ with initial conditions $x(0) = 0.4, y(0) = 0.5$, eigen values are $\lambda_{1,2} = 0.9957 \pm i0.0656$. Hence $|\lambda_{1,2}| = 0.9978 < 1$ satisfied the stability criteria so that the interior fixed point F_2 is stable at $(0.2143, 0.5902)$. The orbit and the phase diagram illustrate the result, see figure - 1.

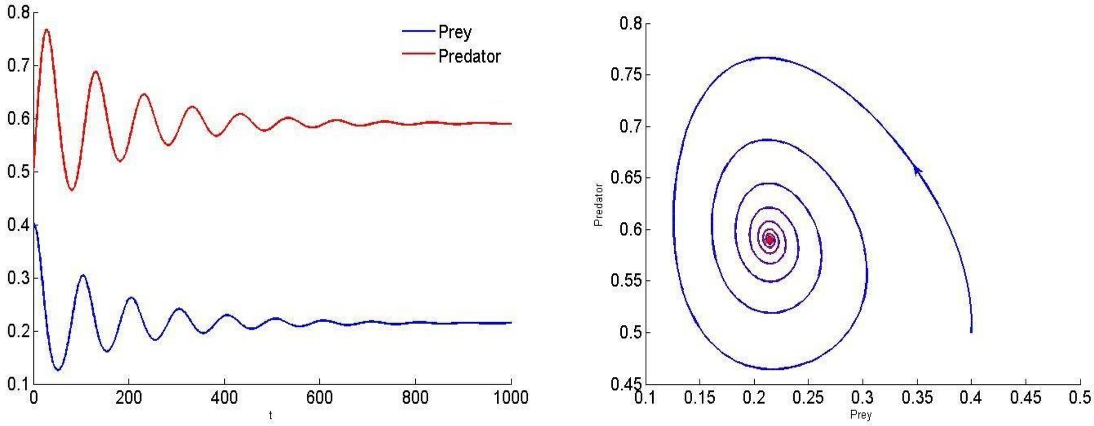

Figure 1: Stable focus of time series and phase portrait for interior fixed point F_2

5. Bifurcations and Sensitive Analysis

The dynamic nature of the system (2) is discussed in this section. Bifurcation diagrams provide information about abrupt changes in the dynamical behavior of the system . The parametric values at which these changes occur are called bifurcation points. They provide information that the dependence of the dynamics on a certain parameter values. Figure 2 shows the dynamic behavior of the system in the absence of harvesting and figure 3 represents the dynamic changes occuring in the system in the presence of harvesting.

In figure 2, the bifurcation diagram is plotted for the parameter values $b = 0.975, c = 0.714, d = 0.534, h = 0.45$ and $\alpha = 0.9$, varying r from 3.5 to 6, time plots of the prey population are presented for $r = 3.67, 4.59, 5.10$ and $r = 5.59$ respectively. When $r = 4.59$, there are period-2 orbits. When $r = 5.10$ there appear period-4 orbits. At last, the 2^n period orbits disappear and the dynamical behavior moves from non-periodic orbits to the chaotic set with the increasing of r.

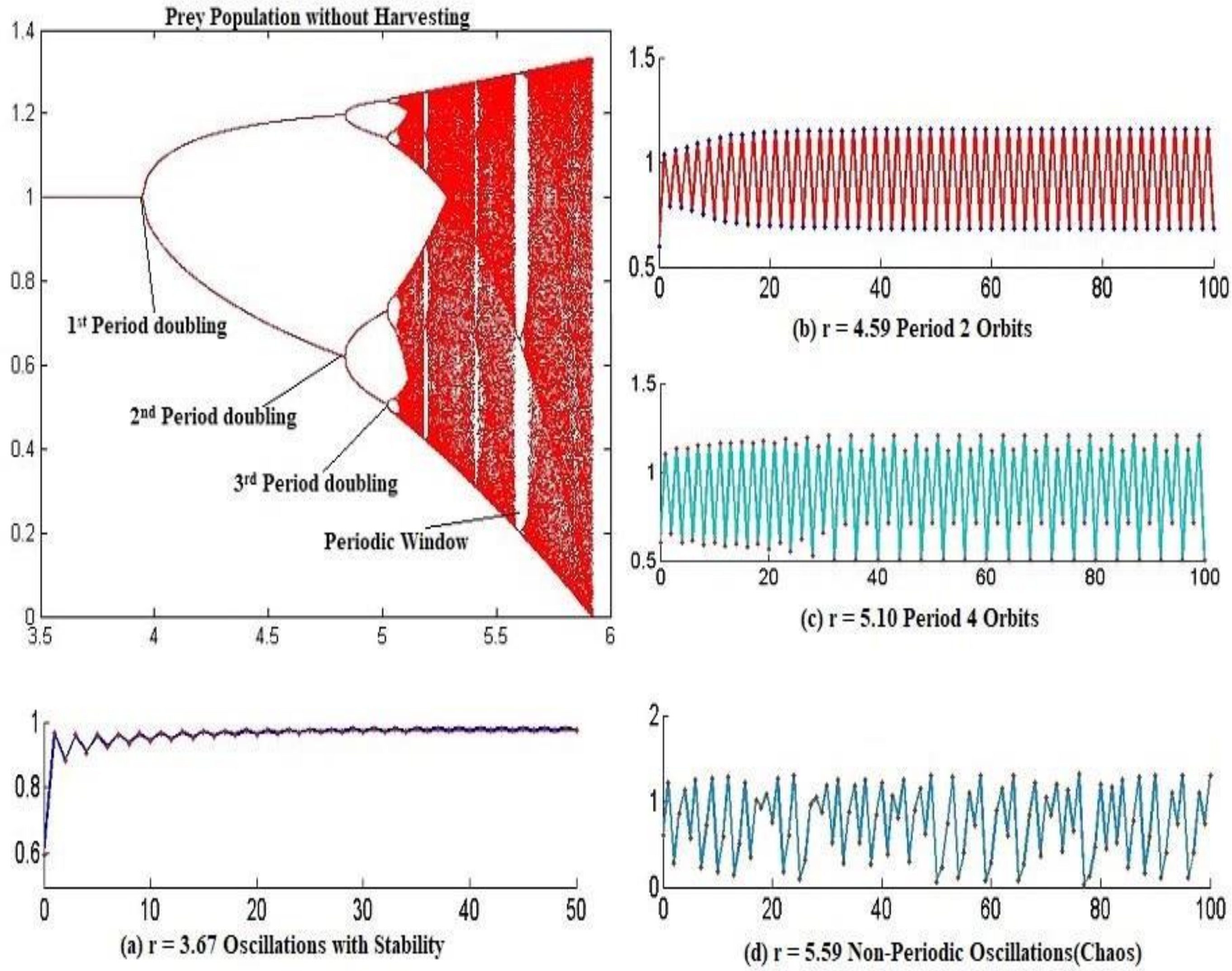

Figure 2: Bifurcation diagram and time plots of the prey population without harvesting

In the presence of harvesting, the bifurcation of prey population for the growth rate *r* occurs in the range [3.5, 6.5], the parameter values are $b = 0.675, c = 0.914, d = 0.234, h = 0.45, \alpha = 0.85$ and $m = 0.1$, as shown in figure 3(a). Local magnifications of periodic windows of fig 3(a) are shown in fig [3(b), 3(c) & 3 (d)]. Hence from the period doubling bifurcation we see that the dynamic behavior of the system changes from stability to chaos, after certain period again it undergoes the process of stability to chaos.

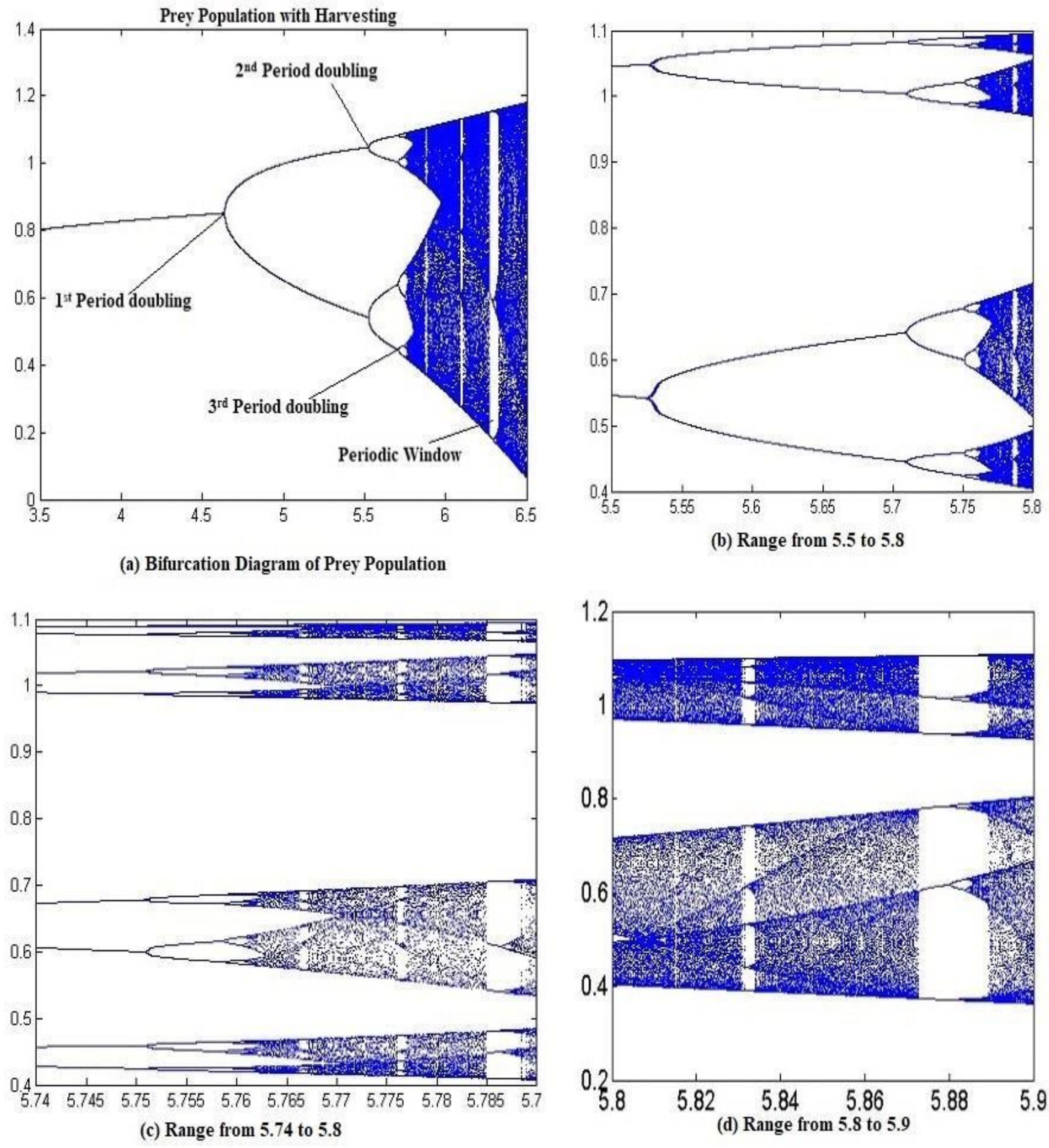

Figure 3: Bifurcation diagram and local magnifications of prey population with harvesting

Sensitive dependence to the initial conditions in the presence and absence of harvesging is shown in figure 4 and figure 5. The sensitivity to initial conditions is a characteristic of chaos. We take two sets of initial values $(x(0), y(0))$ and $(x(0)+0.0001, y(0))$ respectively. The parameter values are (i) $r = 5.46, b = 0.975, c = 0.714, d = 0.534, s = 0.45$ and $\alpha = 0.9$, (ii) $r = 5.64, b = 0.975, c = 0.714, d = 0.534, s = 0.45, \alpha = 0.9$ and $m = 0.09$. The changes occurring in the dynamical behavior of prey population as there is a sensitive change in the initial conditions are plotted in the following figures 4 and 5.

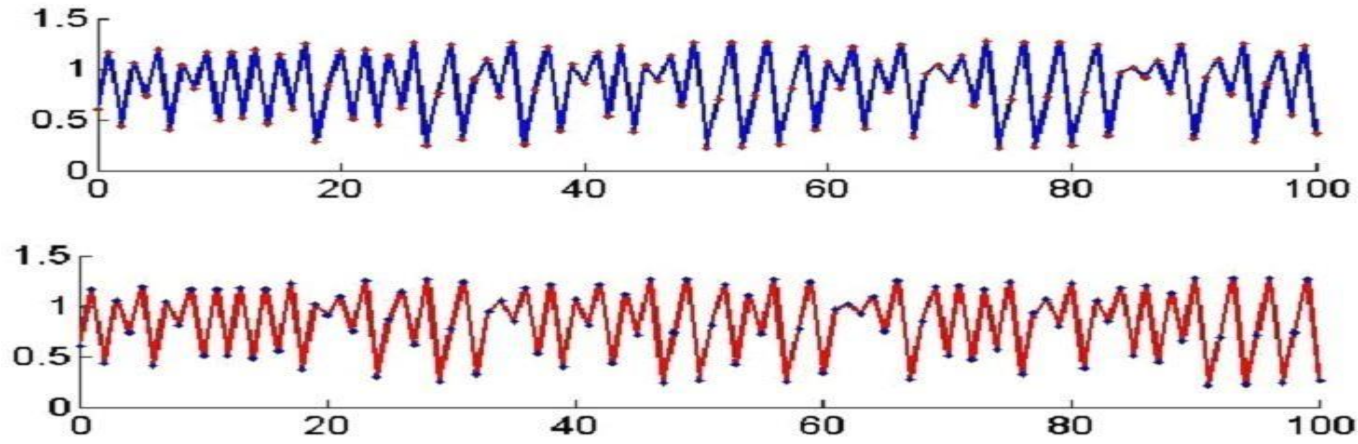

Figure 4: Sensitive dependence to the initial conditions for prey population with harvesting

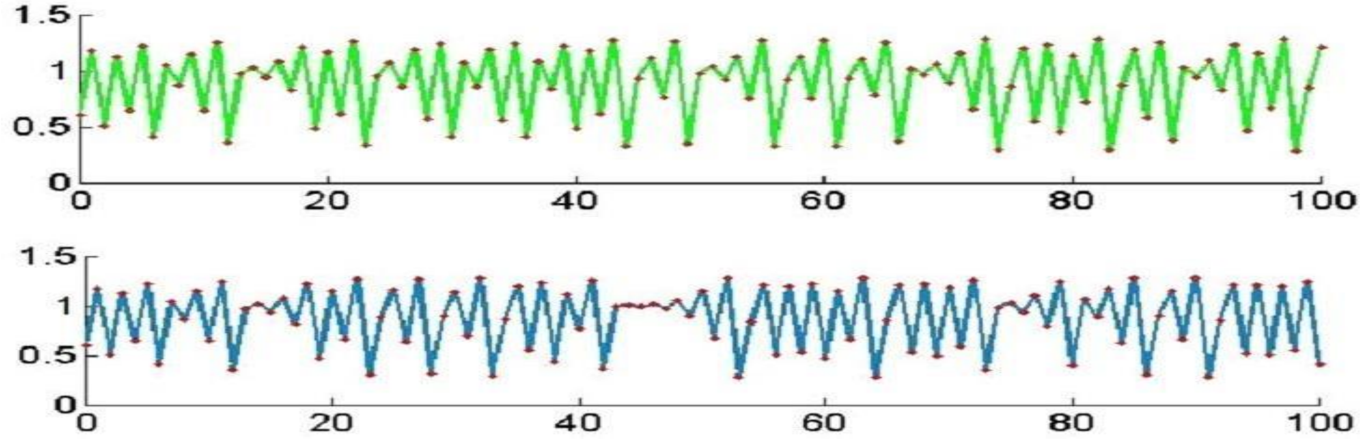

Figure 5: Sensitive dependence to the initial conditions for prey population without harvesting

6. Conclusion

The dynamical behavior of harvesting in a discrete fractional order prey - predator model has been analyzed. Applying a discretization process to the fractional order model, discrete version is obtained. The fixed points are computed and a detailed analysis on the stability of fixed points is carried out with the linerization process and the impact of harvesting on the ecosystem is discussed. A numerical example is presented to illustrate the stability occurring in the system. Bifurcation diagrams are plotted to study the dynamical behavior of the system in the presence/absence of harvesting. The dynamical behavior of the system is found to be sensitive to the initial conditions.

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