A brief note on difference equations

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Abstract: **In this paper we have discussed some theorems for asymptotic stability of linear scalar equations and definitions for linearization of nonlinear difference equations. We have also given appropriate examples with graphs.**

Keywords: **Difference equations, Equilibrium point, Asymptotic stability, Linearization.**

I. INTRODUCTION

The topic of difference equations are interesting and attractive to many mathematicians working in this field. It is a fertile research area. These equations are used to model many real-life phenomena, including economics, biology, etc refer [1,2,3].

Agarwal and Elsayed [4] explored the global stability, periodicity feature, and provided the solution to some special examples of the difference equation. This is just one example of the behaviour of the solution of difference equations that has been studied by numerous researchers.

$$y_{n+1} = a + \frac{dy_{n-l}y_{n-r}}{b - cy_{n-s}}$$

Elabbasy, El-Metwally and Elsayed [5-6] studied the periodicity, boundedness, and global stability of the difference equations, and provided solutions for some particular circumstances

$$y_{n+1} = ay_n - \frac{by_n}{cy_n - dy_{n-l}},$$

$$y_{n+1} = \frac{\propto y_{n-r}}{\beta + \gamma \prod_{i=o}^{r} y_{n-l}}$$

II. Stability Theory

Early work on control systems in mathematics was conducted using differential equations. J. C. Maxwell examined the stability of Watt's flyball governor. He demonstrated a method for linearizing the differential equations of motion in order to derive the system's characteristic equation. By examining how system factors affect stability, he also demonstrated that the system is stable if the roots of the characteristic equation contain negative real portions. E. J. Routh provided a numerical method to determine if the roots of a characteristic equation are stable. Independent of Maxwell, the Russian I. I. Vyshnegradsky examined the stability of regulators using

differential equations. A. B. Stodola used Vyshnegradsky's methods to evaluate the regulation of a water turbine. Unaware of the contributions made by Maxwell and Routh, he assigned A. Hurwitz the task of determining the stability of the

characteristic equation, and Hurwitz came up with an original solution.

III. Jury's Stability Criterion

The behaviour of the linearized system affects the local stability principle for first-order systems or higher-order difference equations. Take into consideration a first order system with n equations,

$$Y_t = (y_{1t}, y_{2t}, y_{3t}, \dots, y_{nt})^T,$$

$$Y_{t+1} = F(Y_t) \tag{3.1}$$

where

$F = (f_1, f_2, f_3, \dots, f_n)^T$ and $f_i = f_i(y_!, y_2, y_3, \dots, y_n)$,

$i = 1,2, \dots, n$. Let us consider system (3.1) has equilibrium point at $\bar{Y}$. Then if $U_t = Y_t - \bar{Y}$, linearisation of system (3.1) about $\bar{Y}$ results to the system

$$U_{t+1} = JU_t \tag{3.2}$$

In (3.2) J represents the Jacobian matrix examined at $\bar{Y}$,

$$J = \begin{pmatrix} \frac{\partial f_1(\bar{Y})}{\partial y_1} & \frac{\partial f_1(\bar{Y})}{\partial y_2} & \dots & \frac{\partial f_1(\bar{Y})}{\partial y_n} \\ \frac{\partial f_2(\bar{Y})}{\partial y_1} & \frac{\partial f_2(\bar{y})}{\partial y_2} & \dots & \frac{\partial f_2(\bar{Y})}{\partial y_n} \\ \vdots & \vdots & \dots & \vdots \\ \frac{\partial f_n(\bar{Y})}{\partial y_1} & \frac{\partial f_n(\bar{Y})}{\partial y_2} & \dots & \frac{\partial f_n(\bar{Y})}{\partial y_n} \end{pmatrix} \tag{3.3}$$

The eigen values of the Jacobian matrix determine the local asymptotic stability of $\bar{Y}$, again that is determined by the existence of partial derivatives in a region containing $\bar{Y}$. So, for $\bar{Y}$ to be locally asymptotically stable, we require that the partial derivatives of f_i should be continuous in an open set containing $\bar{Y}$.

The eigenvalues of the Jacobian matrix are the solutions of the characteristic equation

$$\det(J - \lambda I) = 0$$

Eigenvalues are the roots of the following nth-degree characteristic polynomial :

$$\omega(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_n \qquad (3.4)$$

As the function f_i is real valued, the coefficients in (3.4) are real. For $n = 2, a_1 = -Tr(J)$ and $a_2 = \det(J)$. The Jury conditions or Jury test refers to the requirements that must be met for local asymptotic stability. These conditions make sure that the zeros of the characteristic polynomial (3.4) satisfy

$\lambda_i < 1$.

Theorem 3.1. *If the solutions* $\lambda_i, i = 1,2, \ldots, n$ *of* (3.4), $\omega(\lambda) = 0$ *satisfy* $|\lambda_i| < 1$, *then*

- $\omega(1) = 1 + a_1 + a_2 + \cdots + a_n > 0$,
- $(-1)^n\omega(-1) = 1 - a_1 + a_2 - \cdots + (-1)^n a_n > 0$ (*alternate in sign*),
- $|a_n| < 1$

As a fact a_n is the product of the eigenvalues, $a_n = \lambda_1\lambda_2 \ldots \lambda_n$, so the third condition of theorem 3.1. is followed. There doesn't exist a root $|\lambda_i| \geq 1$, if one of the preceding conditions is not satisfied.

By using the direct approach of Lyapunov, one is able to examine the qualitative properties of the solutions without typically figuring out the solutions themselves. As a result, we view it as one of the key instruments in stability theory. Finding specific real-valued functions with Lyapunov names is necessary for the method to work. Finding the appropriate Lyapunov function for a given equation is the direct method's main drawback. We consider the autonomous difference equation.

Lyapunov's method

The control theory was influenced by A.M. Lyapunov's work. Using a broad conception of energy, he looked into the stability of nonlinear differential equations in 1892. Although his work was used and continued in Russia, sadly, the West was not ready for his beautiful theory, and it was not until about 1960 that its significance was ultimately understood.

$$y(n+1) = f(y(n)) \qquad (3.5)$$

Where $f: \mathbb{R}^r$, $G \subset \mathbb{R}^r$ is continuous. We consider that $\bar{y}$ is an equilibrium point, that is $f(\bar{y}) = \bar{y}$. Let $V: \mathbb{R}^r \to \mathbb{R}$, be a real valued function. The variation of V with respect to (3.5) would be defined as

$$\Delta V(y) = V(f(y)) - V(y) \qquad (3.6)$$

and

$$\Delta V(y(n)) = V(y(n+1)) - V(y(n)) \qquad (3.7)$$

Observe that, if $\Delta V(y) \leq 0$, then V is decreasing along the solution of (3.5). The function V is called a Lyapunov function on a subset H of $\mathbb{R}^r$, if :

- V is continuous on H.
- $\Delta V(y) \leq 0$, whenever y and $f(y)$ is in H.

Theorem 3.2. $\bar{\mathbf{y}}$ *is stable, if V is Lyapunov function for (3.5) in a neighbourhood H of the equilibrium point* $\bar{\mathbf{y}}$ *and V is positive definite with respect to* $\bar{\mathbf{y}}$. *Also,* $\bar{\mathbf{y}}$ *is asymptotically stable if,* $\Delta V(y) < 0$ *wherever* $y, f(y) \in H$ *and* $y \neq \bar{y}$. *In addition,* $\bar{y}$ *is globally asymptotically stable if,* $G = H = \mathbb{R}^k$ *and* $V(y) \to \infty$ *and* $\|y\| \to \infty$.

IV. Asymptotic Stability Theorems of Linear Scalar Equations

Take into account the second order difference equation

$$y_{n+2} + \omega_1 y_{n+1} + \omega_2 y_n = 0 \qquad (4.1)$$

the characteristic equation is

$$\lambda^2 + \omega_1\lambda + \omega_2 = 0 \qquad (4.2)$$

Theorem 4.1. [7] *The condition*

$$1 + \omega_1 + \omega_2 > 0, 1 - \omega_1 + \omega_2 > 0,$$

$$1 - \omega_2 > 0$$

are necessary and sufficient conditions, so that the equilibrium point of equation (4.1) would be asymptotically stable. These conditions can be expressed as

$$|\omega_1| < 1 + \omega_2 < 2$$

Theorem 4.2. [7] *The zero solution of (4.1) is asymptotically stable if and only if*

$$|\omega_1| < 1 + \omega_2 < 2$$

Theorem 4.3. [7] *The zero solution of the third order homogeneous difference equation*

$$y_{n+3} + \omega_1 y_{n+2} + \omega_2 y_{n+1} + \omega_3 y_n = 0 \qquad (4.3)$$

will be stable if and only if

$$|\omega_1 + \omega_3| < 1 + \omega_2, \text{ and } |\omega_2 - \omega_2\omega_3| < 1 - \omega_3^2$$

Consider the rth-order equation

$$y_{n+1} - ay_n + by_{n-r} \qquad (4.4)$$

Theorem 4.4. [7] *Let* a *be a non-negative real number,* b *an arbitrary real number and* r *be a positive integer. The zero solution of (4.4) is asymptotically stable if and only if*

$$|a| < \frac{r+1}{r}$$

and

1. $|a| < b < (a^2 + 1 - 2|a|\cos\phi)^{\frac{1}{2}}$
2. $|b - a| < 1$ and $|b| < (a^2 + 1 - 2|a|\cos\phi)^{\frac{1}{2}}$ *for r even,*

where ϕ *is the solution in* $\left(0, \frac{\pi}{(k+1)}\right)$ *of*

$$\frac{\sin(k\theta)}{\sin(k+1)\theta} = \frac{1}{|a|}$$

Lets get to the general form of the rth-order homogeneous difference equation

$$y_{n+r} + \omega_1 y_{n+r-1} + \omega_2 y_{n+r-2} + \cdots + \omega_{r-1} y_{n+1} + \omega_r y_n \quad (4.5)$$

Theorem 4.5. [7] *The zero solution of (4.5) is asymptotically stable if*

$$\sum_{i=1}^{r} |\omega_i| < 1 \quad (4.6)$$

also the zero solution of this equation is unstable if

$$|\omega_1| - \sum_{i=2}^{r} |\omega_i| > 1 \quad (4.7)$$

V. Linearization of Non-linear Equations

Take into account the $r+1$ order difference equation which is of the form

$$y_{n+1} = f(y_n, y_{n-1}, \dots, y_{n-r}) \quad (5.1)$$

where $f: I^{r+1} \to I$ is a continuous differentiable function and $y_{-r}, y_{-r+1}, \dots, x_0$ are the initial conditions. So there exists a unique solution $\{x_n\}_{n=-r}^{\infty}$ such that $y(-r) = y_{-r}, y(-r+1) = y_{-r+1}, \dots, x(0) = x_0$

Definition 5.1. *An equilibrium point of (5.1) is the point* $\bar{y} \in I$ *such that*

$$f(\bar{y}, \bar{y}, \bar{y}, \dots \bar{y}) = \bar{y}$$

Definition 5.2. *The equilibrium point* $\bar{y}$ *of (5.1) is called locally stable if for any given* $\epsilon > 0$ *there exists* $\delta > 0$ *such that for all* $y_{-r}, y_{-r+1}, \dots, y_{-1}, y_0 \in I$ *if*

$$|y_{-r} - \bar{y}| + |y_{-r+1} - \bar{y}| + |y_0 - \bar{y}| < \delta,,$$

then

$$|y_n - \bar{y}| < \epsilon \text{ for all } n \geq -k$$

Definition 5.3. *The equilibrium point* $\bar{y}$ *of (5.1) is called locally asymptotically stable when x is locally stable and there exists* $\gamma > 0$ *such that for all* $y_{-r}, y_{-r+1}, \dots, y_{-1}, y_0 \in I$ *if*

$$|y_{-r} - \bar{y}| + |y_{-r+1} - \bar{y}| + |y_0 - \bar{y}| < \gamma,$$

then

$$\lim_{n\to\infty} y_n = \bar{y}$$

Definition 5.4. The *equilibrium point* $\bar{y}$ *of (5.1) is said to be global attractor if for all* $y_{-r}, y_{-r+1}, \dots, y_{-1}, y_0 \in I$, *we get*

$$\lim_{n\to\infty} y_n = \bar{y}.$$

Definition 5.5. *When an equilibrium point* x *of (5.1) is locally stable as well as global attractor, then it is called as globally asymptotically stable***.**

We can linearize (5.1) around x if f is continuously differentiable in a region around x. Hence, by chain rule the linearized equation around $\bar{y}$ becomes

$$u_{n+1} = \omega_0 u_n + \omega_1 u_{n-1} + \cdots + w_r u_{n-r} \quad (5.2)$$

where

$$\omega_i = \frac{\partial f}{\partial u_i}(\bar{y}, \bar{y}, \dots, \bar{y})$$

The characteristic equation of (5.2) is given by

$$\lambda^{r+1} - \omega_0 \lambda^r - \omega_1 \lambda^{r-1} - \cdots - \omega_r = 0 \quad (5.3)$$

Example 5.1: Let us observe the difference equation

$$y_{n+1} = y_n^2 - y_n + 1$$

So, $f(y) = y^2 - y + 1$ also by allowing $\bar{y} = \bar{y}^2 - \bar{y} + 1$, We may deduce that there is only one equilibrium point for this equation $\bar{y} = 1$.

We can define equilibrium point graphically as a point in the x-coordinate where the diagonal line $y = x$ and the graph of f intersects each other.

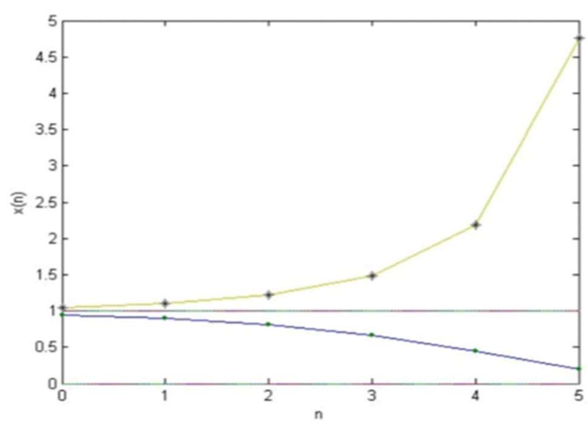

Fig. 1. Equilibrium points of $y_{n+1} = y_n^2 - y_n + 1$

Example 5.2: The equation

$$y_{n+1} = y_n^3$$

has three fixed points. Figure 1.2 below illustrates this, and they are $\bar{y} = -1,0,1$.

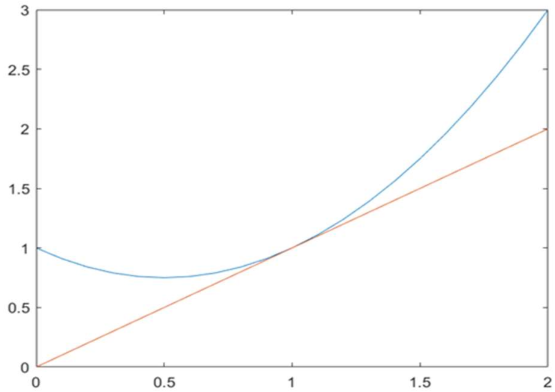

Fig. 2. Equilibrium points of $y_{n+1} = y_n^3$

Analysing the behaviour of a dynamical system's solutions close to an equilibrium point is one of the fundamental goals of dynamical system study. The stability theory is the name of this subject.

Example 5.3: Let us take the first order difference equation

$$y_{n+1} = \frac{1}{2}y_n - 1$$

then the equilibrium point of the function $f(y) = \frac{1}{2}y - 1$ is the point $y = -2$, hence the fixed point of our difference equation is $\bar{y} = -2$. If we use the equation's beginning condition as $y_0 = 1$, following that as the figure 1.3makes it clear, In addition to being a stable point, $\bar{y} = -2$ is also asymptotically stable.

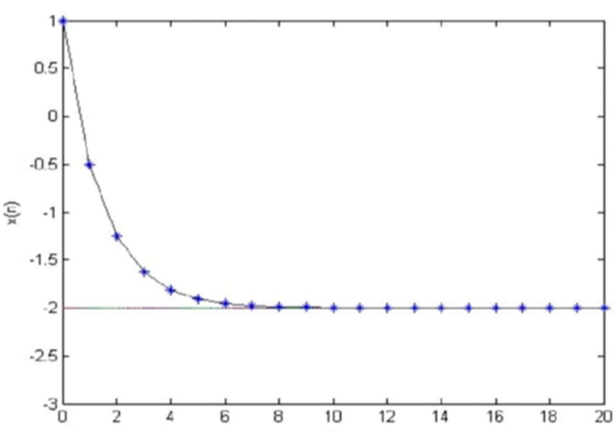

Fig. 3. The stability of $\bar{y} = -2$ of $y_{n+1} = \frac{1}{2}y_n - 1$

Example 5.4: Consider the difference equation

$$y_{n+1} = y_n^2$$

The equilibrium points are $\bar{y} = 0, \bar{y} = 1$. It is clear from the figure 1.4

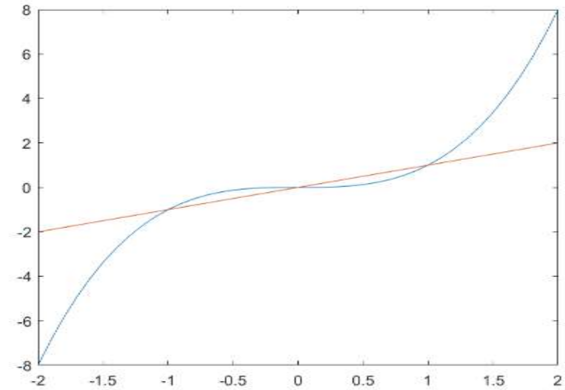

Fig. 4. The stability of $\bar{y} = 1$ of $y_{n+1} = y_n^2$

that $\bar{y} = 1$ is unstable equilibrium point.

Example 5.5: Consider the non-linear difference equation

$$y_{n+2} = 2y_{n+1}^2 - \frac{1}{y_n}$$

The equilibrium point can be located using

$$\bar{y} = 2\bar{y}^2 - \frac{1}{y}$$

This could be expressed as

$$2\bar{y}^3 - \bar{y}^2 - 1 = 0$$

which is

$$(\bar{y} - 1)(2\bar{y}^2 + \bar{y} + 1) = 0$$

So, this equation's equilibrium point is $\bar{x} = 1$..

Take $f(p,q) = 2p^2 - \frac{1}{q}$ then,

$$\frac{\partial f}{\partial p} = 4p, \frac{\partial f}{\partial q} = \frac{1}{q^2}$$

Therefore, the linearized equation we use to describe the equilibrium point $\bar{y} = 1$ is

$$z_{n+2} = 4z_{n+1} + z_n$$

which can be written as

$$z_{n+2} - 4z_{n+1} - z_n = 0$$

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